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Sensitivity of the Three-Body Calculations to Different Effects

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The bound state of few-body systems in light nuclei is studied as a three-body problem. The three-body problem is solved following the different approaches of the Faddeev formalism as well as the unitary pole approximation. Separable approximations are introduced to reduce the three-body problem to a set of coupled integral equations. Numerical calculations are carried out for the resulting integral equations and the separable expansion. In the present work, we calculate the ground-state binding energy of the bound three-nucleon system ^{3}H. The main interest of the present work is to investigate the sensitivity of the three-body binding energy to different effects in the problem. For this reason, we study the dependence of the three-body binding energy of different forms of local and separable two-body potentials, on the effective range of the two-body potentials, and on the percent of the D state in the deuteron wave function. Also, we test the sensitivity of the three-body binding energy to the considered number of terms from the separable expansion.

1. INTRODUCTION

The three-body problem has been proved a powerful tool in studying the static properties of light nuclei. Faddeev (1960) successfully obtained an exact solution for the three-body problem leading to a well-behaved set of three-body equations involving the two-body T matrix rather than the potential. The T matrix in this approach plays a central role. In other words, the two-body T matrix in the three-body Faddeev equations plays the part of a potential in the two-body Lippmann–Schwinger equation. Using separable potentials, the Faddeev equations reduce to coupled integral equations in one continuous variable. When using local potentials,

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the Faddeev equations are reduced to equations in two continuous variables. Different separable approximations and separable expansions have been suggested by different authors (Mitra, 1962; Amado, 1963; Lovelace, 1964; Kowalski, 1965; Noyes, 1965; Fuda, 1969; Kharchenko and Petrov, 1969) for the T matrix. The Faddeev equations were generalized to be applicable for the multiparticle scattering problem (Weinberg, 1964; Sugar and Blankenbecler, 1964; Rosenberg, 1965) and for the many-cluster problem (Osman, 1977) by constructing N-body integral equations with square integrable kernels. Also, Brady et al. (1969) introduced a unitary pole approximation, using the two-body binding energy and wave function in determining the form factor for the unitary pole approximation separable T matrix.

In the present work, we consider the case of the three-nucleon system ^{3}H. This nucleus is studied as a three-body problem. We solve the ^{3}H nucleus problem following two different approaches. The first approach is the Faddeev formalism, using the separable approximation (Brayshaw, 1968, 1969; Ball et al., 1968; Beam, 1969; Chen and Ishihara, 1969; Chen et al., 1969). The second approach is the unitary pole approximation. We use both approaches here as a matter of comparison (Kharchenko et al., 1968; Phillips, 1968; Kok et al., 1968; Levinger, 1969). Direct numerical calculations have been carried out for the resulting integral equations for the three-body binding energy of ^{3}H. For the nucleon–nucleon interactions we use both local potentials and nonlocal separable potentials. The two-nucleon local potentials used are taken to have square well, exponential, Yukawa, and also Hulthén forms. The separable nucleon–nucleon interactions used are taken to have potential functions of the Yamaguchi, Gaussian, Tabakin, Mongan, and Reid forms. The different parameters of all local and separable nucleon–nucleon potentials are determined in such a way that they fit the same two-body phase shifts.

The aim of the work is to investigate different effects in the three-body calculations. We study the difference obtained in the three-nucleon ^{3}H binding energy by using local or separable potentials fitting the same two-body phase shifts. We also test the effect of changing the singlet effective range on the binding energy. We estimated the dependence of the three-nucleon ^{3}H binding energy on the percent D state of the deuteron wave function. To see the convergence of the separable expansion, we investigate the effect and contribution of the different partial wave expansion terms on the binding energy. Naturally, since in the present calculations we use the Faddeev formalism as well as the unitary pole approximation, we obtain the difference between these approaches by extracting the binding energy. Then we calculate the triton ^{3}H binding energy using singlet two-nucleon potentials with a Yamaguchi form factor

of a varying parameter defining the shape of the two-nucleon interaction with an estimate of the resulting change in the binding energy.

The investigation of all these effects on the three-nucleon ^{3}H binding energy is studied in the present work. From all the results we can estimate the sensitivity of the three-body binding energy to changes in the different factors considered here.

In Section 2, we introduce the different formulas of the approximations considered. Numerical calculations and results are given in Section 3. Discussion and conclusions are presented in Section 4.

2. FORMULAS OF THE DIFFERENT APPROXIMATIONS

In the present work, the three-body problem is solved following two different approaches. The first is by following the Faddeev formalism. The three-body Faddeev equations are converted into an infinite set of coupled integral equations. This infinite set of coupled integral equations is reduced to a finite and well-behaved set of coupled integral equations by using separable two-body potentials. The three particles in the three-body system are denoted by the symbols i, j, k in a cyclic permutation. The three-body scattering operator from the Faddeev equations can be written in the form

$$T(z)=\sum_{i=1}^{3} T^{(i)}(z)+\sum_{i,j=1}^{3} T^{(i)}(z)\Psi_{ij}(z)T^{(j)}(z), \tag{2.1}$$

where $T^{(i)}(z)$ is the two-body scattering operator of particles j and k. $\Psi_{ij}(z)$ is the operator describing the transition of the system from an initial scattering state of particles i and k to a final scattering state of particles j and k. This operator satisfies the equation

$$\Psi_{ij}(z)=(\delta_{ij}-1)G_0(z)+\sum_{k=1}^{3}(\delta_{ik}-1)G_0(z)T^{(k)}(z)\Psi_{kj}(z), \tag{2.2}$$

where $G_0(z)$ is the free three-body propagator. The T matrices can be written as

$$\langle \mathbf{p}_i', \mathbf{q}_i' | T^{(i)}(z) | \mathbf{p}_i, \mathbf{q}_i \rangle = \delta(\mathbf{q}_i'-\mathbf{q}_i)\langle \mathbf{p}_i' | t^{(i)}(z-q_i^2) | \mathbf{p}_i \rangle \tag{2.3}$$

where $t^{(i)}(z)$ is the two-body T matrix. $\mathbf{p}_i$ is the relative momentum of particles j and k, and $\mathbf{q}_i$ is the momentum of particle i relative to the center of mass of particles j and k. In equation (2.3), if $t^{(i)}(z)$ has a bound-state pole at $z=-\varepsilon_i$, then the three-body T matrix $T^{(i)}(z)$ will have a branch

point there, with a cut going from $-\varepsilon_i$ to $+\infty$, which is the right-hand cut. The two-body T matrices are written in a separable expansion as

$$\langle \mathbf{p}'|t^{(i)}(z-q_i^2)|\mathbf{p}\rangle = \sum_{\lambda=1}^{N_i} \frac{g_\lambda^{(i)}(\mathbf{p}')g^{*(i)}_\lambda(\mathbf{p})}{D_\lambda^{(i)}(z-q_i^2)} \tag{2.4}$$

The function $g_\lambda^{(i)}(\mathbf{p})$ is a vertex form factor referring to a formation of a two-body bound state or a resonant state. The denominator function $D_\lambda^{(i)}(z-q_i^2)$ vanishes when $z=-\varepsilon_i$, which corresponds to a bound or resonant state of the particles j and k with energy $-\varepsilon_i$. Then, the set of three-body coupled integral equations is given by

$$\Psi_{\lambda\mu}^{l'l}(q',q;z)=\Phi_{\lambda\mu}^{l'l}(q',q;z) + \tfrac{1}{2}\sum_{\lambda'}\int_0^\infty dq''q''^2 \frac{\Phi_{\lambda\lambda'}^{l'l}(q',q'';z)\Psi_{\lambda'\mu}^{l'l}(q'',q;z)}{D\left(z-\frac{3}{4}q''^2\right)} \tag{2.5}$$

where $\Psi(q',q;z)$ in equation (2.5) is the three-body T-matrix element for a final state of two particles in a relative l state and the third particle in an l state relative to the center of mass of the first two. $\Phi(q',q;z)$ is the inhomogeneous term which is a symmetrized T matrix with one noninteracting particle. In the present calculations, we restrict ourselves to a finite set of values of l as $0\leqslant l\leqslant L$. The numerical calculations are carried out with values of L and N_i which are not too large.

The unitary pole approximation is suggested by Brady et al. (1969). They introduced this approximation by first considering a knowledge of the two-body binding energy and ground-state wave function. Then, a separable potential is constructed with identical binding energy and wave function. This separable potential is used to find the off-shell values of the two-body T matrix and, consequently, the three-body binding energy. In this way, the unitary pole approximation separable potential could be written in the form

$$\langle p|V_{ij}|q\rangle = -\lambda_n f_n(p)f_n(q) \tag{2.6}$$

where the form factor $f_n(p)$ and the potential strength λ_n are unknowns and can be found by solving the corresponding Schrödinger equation for the given two-body binding energy ε and ground-state wave function $|\Phi\rangle$. Then the two-body T matrix is given (Brady et al., 1969) in a separable form, which is used to determine the three-body binding energy.

3. NUMERICAL CALCULATIONS AND RESULTS

The three-body binding energy for the three-nucleon system ^{3}H is obtained by numerical solution of the resulting three-body integral equations. In the present work, we are interested in investigating two-body interaction effects on the three-body binding energy. The difference obtained in the three-nucleon system ^{3}H binding energy by using local potentials or separable potentials is studied. The effect on the binding energy of changing the singlet effective range is tested. Also, we study the dependence of the three-nucleon ^{3}H binding energy on the percent D state of the deuteron wave function. We also investigate the contribution of the different partial wave expansion terms on the binding energy, to test the convergence of the separable expansion. Also, the three-nucleon ^{3}H binding energy is calculated using the unitary pole approximation so as to compare it with those obtained following the Faddeev formalism. When using singlet two-nucleon potentials with a Yamaguchi form factor, we calculate the triton ^{3}H binding energy with a form factor of varying parameter, which defines the shape of the two-nucleon interaction.

In the present numerical calculations, the three-body ground-state energy for the three-nucleon ^{3}H system is calculated with very high accuracy. With the features of the potentials used, in our calculations we apply the well-behaved Schmidt–Hilbert (Courant and Hilbert, 1953) theory of integral equations. In the numerical calculations of the three-body integral equations, a 36-point Gaussian integration is used. The integral equation is converted into an eigenvalue equation, maintaining a rapid convergence at infinity. These eigenvalues are given as a function of the energy z. The three-body bound state energies are those values of the energy z for which a matrix eigenvalue takes the value 1.

Direct numerical calculations for the resulting three-body Faddeev integral equations are carried out to calculate the binding energy of the three-nucleon ^{3}H system, using local and separable potentials for the two-nucleon interactions. The binding energy is calculated also by using the unitary pole approximation. The local potentials used are taken to have square well, exponential, Yukawa, and also Hulthén forms. The separable interactions are taken to have a potential function of the Yamaguchi, Gaussian, Tabakin, Mongan, and Reid forms. The values obtained of the binding energy of ^{3}H nucleus are given in Table I. The binding energies are also calculated for different values of the singlet effective range r_0, and for different values of the percent D state of the deuteron wave function. The results obtained are introduced in Table II. The first three terms of the separable expansion are calculated separately to estimate the contribution of each term to the binding energy, which at the same time is also a test of

TABLE I. Binding Energies for ^{3}H Nucleus in (MeV), for Different Potentials, $r_0 = 2.70$ fm and $P_D = 4\%$

Nucleon–nucleon potentials	Forms	Binding energies in (MeV) Three-body Faddeev integral equations	Unitary pole approximation
local	square well	9.58	9.73
	exponential	10.21	10.39
	Yukawa	10.45	10.59
	Hulthén	9.95	9.98
separable	Yamaguchi	8.83	9.05
	Gaussian	8.67	8.93
	Tabakin	8.86	8.98
	Mongan	8.51	8.76
	Reid	8.54	8.81
The experimental value		8.48	

TABLE II. Binding energies for ^{3}H Nucleus in (MeV), for Different Values of r_0 (fm) and P_D(%), for a Potential of the Yamaguchi Form

P_D (%) \ r_0(fm)	2.25	2.40	2.55	2.70	2.85
1.0	10.76	10.31	9.93	9.68	9.26
2.5	10.28	9.89	9.48	9.27	8.74
4.0	9.72	9.36	9.01	8.83	8.28
5.5	9.36	8.84	8.59	8.38	7.86
7.0	8.83	8.31	8.12	7.85	7.39

TABLE III. Binding Energies for ^{3}H Nucleus in (MeV), for Individual Different Terms of the Partial Wave Expansion

Nucleon–nucleon potential	Forms	Partial wave expansion values l		
		0	1	2
local	square well	8.494	0.987	0.099
	exponential	9.022	1.084	0.114
	Yukawa	9.236	1.093	0.121
	Hulthén	8.840	1.002	0.108
separable	Yamaguchi	7.762	0.981	0.087
	Gaussian	7.723	0.862	0.085
	Tabakin	7.849	0.923	0.088
	Mongan	7.531	0.895	0.084
	Reid	7.545	0.911	0.084

TABLE IV. Binding Energies for ^{3}H Nucleus in (MeV), for Different n Values (Different Shapes of Potentials), $r_0 = 2.70$ fm

	Different values of n		
The potential form	1	2	3
$(p^2+a^2)^{-n}$	8.83	8.41	8.25

the convergence of the expansion. These results are presented in Table III. Also, we carried out calculations for the binding energies of ^{3}H nucleus using singlet potentials with form factors given as

$$g(p)=(p^2+a^2)^{-n} \tag{3.1}$$

for different values of n. The parameter n defines the shape of the two-nucleon interaction and allows for the finiteness of the range of internucleon forces. The case with $n=0$ is the zero-range force potential. If $n=1$ in expression (3.1), this is the Yukawa potential; if $n=1$ in the momentum space, it is the Yamaguchi singlet shape. If $n=2$ in expression (3.1), it gives the exponential potential and so on. We calculate the binding energies for ^{3}H nucleus with the potential given by expression (3.1) for different values of n, which means different shapes of the two-nucleon interactions (keeping the singlet effective range parameters fixed). The results obtained are given in Table IV.

4. DISCUSSION AND CONCLUSIONS

In the present calculations we deduced one of the most important physical observables of the nuclei, the binding energy. We calculated the binding energy of the three-nucleon system ^{3}H using different local and separable forms for the nucleon–nucleon interactions. For the local potentials we use square well, exponential, Yukawa, and Hulthén potential forms. When using separable potentials, we used potential functions of the Yamaguchi, Gaussian, Tabakin, Mongan, and Reid forms. In the present calculations, we followed two methods. One method is by direct numerical calculations of the resulting coupled integral equations from the Faddeev formalism using separable expansion, the other is by using the unitary pole approximation. Here we calculate the binding energy of the ^{3}H nucleus, changing values of different two-nucleon parameters to investigate the dependence and sensitivity of the ^{3}H three-body binding energy on different effects. All the results obtained are introduced in Tables I–IV. From

these tables we can conclude the following:

(i) Separable and local potentials are taken to fit the same two-body phase shifts. In spite of that, the extracted three-nucleon ^{3}H binding energies using local potentials differ by about 1–1.5 MeV from the three-nucleon ^{3}H binding energies using separable potentials. The separable potentials always give the lower values.

(ii) The three-nucleon ^{3}H binding energies calculated following the Faddeev formalism always differ by about 0.2–0.3 MeV from that calculated by the unitary pole approximation using the same two-nucleon potentials. This difference on the average is about 2.5%. The Faddeev formalism always gives the lower values.

(iii) Each step of changing the singlet effective range r_0 results in changing the three-nucleon ^{3}H binding energy by about 0.4 MeV on the average. This means that a change in the singlet effective range r_0 on the average of 5.88% gives a change in the binding energy on the average of 2.94%. As a function of r_0, the binding energy always decreases by increasing the singlet effective range.

(iv) Each step of changing the percent of the D state in the deuteron wave function P_D results in changing the three-nucleon ^{3}H binding energy by about 0.5 MeV on the average. This means that a change in the percent of the D state in the deuteron wave function P_D on the average of 75% gives a change in the binding energy on the average of 4.94%. As a function of P_D, the binding energy always decreases by increasing the percent of the D state in the deuteron wave function P_D. The value of P_D is always required to be $P_D \geqslant 0.45\%$ as a requirement (Levinger, 1969) for the value of the deuteron quadrupole moment

(v) In studying the contribution of each term in the partial wave expansion, it is found that the first term for $l=0$ contributes to the three-nucleon ^{3}H binding energy by about 87.90%; the second term with $l=1$ contributes by about 11.11%; while the contribution to the three-nucleon binding energy from the third term with $l=2$ is only of about 0.99%.

(vi) For Yamaguchi potentials with form factors given by equation (3.1), the three-nucleon binding energy decreases by increasing the parameter n, which changes the shape of the two-body potential, keeping the singlet effective range parameters fixed. If n is increased from 1 to 2, the binding energy is decreased by about 0.4 MeV, which is about 4.87%. Increasing n from 2 to 3, the binding energy decreases further by about 0.2 MeV, which is a further 1.92% approximately. Increasing n from 1 to 3 makes the three-nucleon binding energy decrease by about 0.6 MeV, which means that it decreases by about 6.79%.

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